This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06644503 6

Ma

MATTER AND MOTION.

BY

J. CLERK MAXWELL,

M.A., LL.D., Edin., F.R.SS. L. & E.,

Honorary Fellow of Trinity College, and
Professor of Experimental Physics
in the University of Cambridge.

REPRINTED FROM VAN NOSTRAND'S MAGAZINE.

NEW YORK:
D. VAN NOSTRAND, PUBLISHER,
23 MURRAY AND 27 WARREN STREET.
1878.

REPAIR No. 5 8 0 '08

REPAI[illegible] '08

1024

ASTOR LIBRARY NEW-YORK OCT 9 1818

PREFACE TO AMERICAN EDITION

THE eminent position of the author of this essay, and the importance which the questions here discussed have recently assumed, seemed to demand an increase of facilities for obtaining the work in this country.

The interest manifested in the discussions of kindred subjects in VAN NOSTRAND'S MAGAZINE, induced the publisher to lay this little treatise before the readers of that periodical. It was, therefore, published in the April and May numbers of the Magazine, and is here re-printed entire, with some changes in mechanical arrangement and execution of cuts and typography from the foreign edition, which are believed to be much to the advantage of the reader.

The importance of such clear conceptions of the fundamental principles of physics as are afforded by so eminent a writer is unquestionable.

CONTENTS.

CHAPTER I.

Nature of Physical Science 9
Definition of a Material System 11
Definition of Internal and External 11
Definition of Configuration 12
Diagrams 12
A Material Particle 13
Relative Position of Two Material Particles 14
Vectors 15
System of Three Particles 16
Addition of Vectors 17
Subtraction of One Vector from another 18
Origin of Vectors 19
Relative Position of Two Systems 20
Three Data for the Comparison of Two Systems 21
On the Idea of Space 23
Error of Descartes 25
On the Idea of Time 27
Absolute Space 29
Statement of the General Maxim of Physical Science. 30

CHAPTER II.

ON MOTION.

Definition of Displacement 32
Diagram of Displacement 33
Relative Displacement 35
Uniform Displacement 37
On Motion 38
On the Continuity of Motion 39
On Constant Velocity 40
On the Measurement of Velocity when Variable 41
Diagram of Velocities 43
Properties of the Diagram of Velocities 44
Meaning of the Phrase "at Rest" 46
On Change of Velocity 47
On Acceleration 48
On the Rate of Acceleration 49
Diagram of Accelerations 51
Acceleration a Relative Term 51

CHAPTER III.

ON FORCE.

Kinematics and Kinetics..... 5
Mutual Action between Two Bodies—Stress..... 5
External Force..... 5
Different Aspects of the same Phenomenon..... 5
Newton's Laws of Motion..... 5
The First Law of Motion..... 5
On the Equilibrium of Forces..... 5
Definition of Equal Times..... 6
The Second Law of Motion..... 6
Definition of Equal Masses and of Equal Forces..... 6
Measurement of Mass..... 6
Numerical Measurement of Force..... 6
Simultaneous Action of Forces on a Body..... 7
On Impulse..... 7
Relation between Force and Mass..... 7
On Momentum..... 7
Statement of the Second Law of Motion in Terms of Impulse and Momentum..... 7
Addition of Forces..... 7
The Third Law of Motion..... 7
Action and Reaction are the Partial Aspects of a Stress 7
Attraction and Repulsion..... 7
The Third Law True of Action at a Distance..... 8
Newton's Proof not Experimental..... 8

CHAPTER IV.

ON THE PROPERTIES OF THE CENTER OF MASS OF A MATERIAL SYSTEM.

Definition of a Mass-Vector..... 8
Center of Mass of Two Particles..... 8
Center of Mass of a System..... 8
Momentum Represented at the rate of Change of a Mass-Vector..... 8
Effect of External Forces on the Motion of the Center of Mass..... 8
The Motion of the Center of Mass of a System is not Affected by the Mutual Action of the Parts of the System..... 9
First and Second Laws of Motion..... 9
Method of Treating Systems of Molecules..... 9
By the Introduction of the Idea of Mass we pass from Point-Vectors, Point-Displacements, Velocities, Total Accelerations, and Rates of Acceleration, to Mass-Vectors, Mass-Displacements, Momenta, Impulse and Moving Forces..... 9
Definition of a Mass-Area..... 9
Angular Momentum..... 9
Momentum of a Force about a Point..... 9
Conservation of Angular Momentum..... 9

CHAPTER V.

ON WORK AND ENERGY.

itions..... 101
ciple of Conservation of Energy..... 101
ral Statement of the Principle of the Conserva-
ion of Energy..... 103
surement of Work..... 104
ntial Energy..... 107
etic Energy..... 108
que Forces..... 109
etic Energy of Two Particles Referred to their
Center of Mass..... 114
etic Energy of a Material System Referred to its
Center of Mass..... 115
liable Kinetic Energy..... 118
ential Energy..... 120
sticity..... 122
ion at a Distance..... 123
ory of a Potential Energy more Complicated
than that of Kinetic Energy..... 125
lication of the Method of Energy to the Calcula-
tion of Forces..... 126
rification of the Direction of Forces..... 129
lication to a System in Motion..... 130
lication of the Method of Energy to the Investiga-
tion of Real Bodies..... 131
iables on which the Energy Depends..... 132
rgy in Terms of the Variables..... 134
ory of Heat..... 135
d a Form of Energy..... 136
rgy Measured as Heat..... 137
ntific Work to be Done..... 139
tory of the Doctrine of Energy..... 139
the Different Forms of Energy..... 142

CHAPTER VI.

RECAPITULATION.

rospect of Abstract Dynamics..... 146
ematics..... 146
n..... 147
es..... 148
tivity of Dynamical Knowledge..... 149
tivity of Force..... 150
tion..... 153
ton's Determination of the Absolute Velocity of
Rotation..... 156
ault's Pendulum..... 160
er and Energy..... 163
of a Material Substance..... 165
gy not Capable of Identification..... 166

Absolute Value of the Energy of a Body Unknown... 166
Latent Energy..... 167
A Complete Discussion of Energy would include the whole of Physical Science..... 168

CHAPTER VII.

THE PENDULUM AND GRAVITY.

On Uniform Motion in a Circle..... 169
Centrifugal Force..... 171
Periodic Time..... 172
On Simple Harmonic Vibrations..... 173
On the Force Acting on the Vibrating Body..... 173
Isochronous Vibrations..... 175
Potential Energy of the Vibrating Body..... 176
The Simple Pendulum..... 177
A Rigid Pendulum..... 180
Inversion of the Pendulum..... 183
Illustrations of Kater's Pendulum..... 185
Determination of the Intensity of Gravity..... 186
Method of Observation..... 187
Estimation of Error..... 190

CHAPTER VIII.

UNIVERSAL GRAVIT TION.

Newton's Method..... 192
Kepler's Laws..... 193
Angular Velocity..... 194
Motion about the Center of Mass..... 195
The Orbit..... 196
The Hodograph..... 196
Kepler's Second Law..... 197
Force on a Planet..... 198
Interpretation of Kepler's Third Law..... 202
Law of Gravitation..... 202
Amended Form of Kepler's Third Law..... 204
Potential Energy due to Gravitation..... 205
Kinetic Energy of the System..... 207
Potential Energy of the System..... 209
The Moon is a Heavy Body..... 210
Cavendish's Experiment..... 212
The Torsion Balance..... 213
Method of the Experiment..... 216
Universal Gravitation..... 218
Cause of Gravitation..... 220
Application of Newton's Method of Investigation..... 222
Methods of Molecular Investigations..... 223
Importance of General and Elementary Properties... 224

MATTER AND MOTION.

CHAPTER I.

INTRODUCTION.

NATURE OF PHYSICAL SCIENCE.—Physical science is that department of knowledge which relates to the order of nature, or, in other words, to the regular succession of events.

The name of physical science, however, is often applied in a more or less restricted manner to those branches of science in which the phenomena considered are of the simplest and most abstract kind, excluding the consideration of the more complex phenomena, such as those observed in living beings.

The simplest case of all is that in which an event or phenomenon can be described as a change in the arrangement of certain bodies. Thus the motion

of the moon may be described by stat
the changes in her position relative
the earth in the order in which they
low one another.

In other cases we may know that sc
change of arrangement has taken pl
but we may not be able to ascert
what that change is.

Thus when water freezes we kn
that the molecules, or smallest parts
the substance, must be arranged differe
ly in ice and in water. We also kn
that this arrangement in ice must hav
certain kind of symmetry, because
ice is in the form of symmetrical cryst
but we have as yet no precise knowled
of the actual arrangement of the m
cules in ice. But whenever we can cc
pletely describe the change of arran
ment we have a knowledge, perfect
far as it extends, of what has taken pla
though we may still have to learn
necessary conditions under which a s
ilar event will always take place.

Hence the first part of physical scie
ralates to the relative position a
motion of bodies.

Definition of a Material System.—In all scientific procedure we begin by marking out a certain region or subject as the field of our investigations. To this we must confine our attention, leaving the rest of the universe out of account till we have completed the investigation in which we are engaged. In physical science, therefore, the first step is to define clearly the material system which we make the subject of our statements. This system may be of any degree of complexity. It may be a single material particle, a body of finite size, or any number of such bodies, and it may even be extended so as to include the whole material universe.

Definition of Internal and External.—All relations or actions between one part of this system and another are called Internal relations or actions.

Those between the whole or any part of the system, and bodies not included in the system, are called External rela-

tions or actions. These we study only so far as they affect the system itself, leaving their effect on external bodies out of consideration. Relations and actions between bodies not included in the system are to be left out of consideration. We cannot investigate them except by making our system include these other bodies.

DEFINITION OF CONFIGURATION.—When a material system is considered with respect to the relative position of its parts, the assemblage of relative positions is called the configuration of the system.

A knowledge of the configuration of the system at a given instant implies a knowledge of the position of every point of the system with respect to every other point at that instant.

DIAGRAMS.—The configuration of material systems may be represented in models, plans, or diagrams. The model or diagram is supposed to resemble the

material system only in form, not necessarily in any other respect.

A plan or a map represents on paper in two dimensions what may really be in three dimensions, and can only be completely represented by a model. We shall use the term Diagram to signify any geometrical figure, whether plane or not, by means of which we study the properties of a material system. Thus, when we speak of the configuration of a system, the image which we form in our minds is that of a diagram, which completely represents the configuration, but which has none of the other properties of the material system. Besides diagrams of configuration we may have diagrams of velocity, of stress, &c., which do not represent the form of the system, but by means of which its relative velocities or its internal forces may be studied.

A Material Particle.—A body so small that, *for the purposes of our investigation*, the distances between its

different parts may be neglected, is called a material particle.

Thus in certain astronomical investigations the planets, and even the sun, may be regarded each as a material particle, because the difference of the actions of different parts of these bodies does not come under our notice. But we cannot treat them as material particles when we investigate their rotation. Even an atom, when we consider it as capable of rotation, must be regarded as consisting of many material particles.

The diagram of a material particle is of course a mathematical point, which has no configuration.

Relative Position of two Material Particles.—The diagram of two material particles consists of two points, as, for instance, A and B.

The position of B relative to A is indicated by the direction and length of the straight line $\overline{A\,B}$ drawn *from* A *to* B. If you start from A and travel in the direction indicated by the line $\overline{A\,B}$ an

for a distance equal to the length of that line, you will get to B. This direction and distance may be indicated equally well by any other line, such as $\overline{a\ b}$, which is parallel and equal to $\overline{A\ B}$. The position of A with respect to B is indicated by the direction and length of the line $\overline{B\ A}$, drawn *from* B *to* A, or the line $\overline{b\ a}$, equal and parallel to $\overline{B\ A}$.

It is evident that $\overline{B\ A} = -\overline{A\ B}$.

In naming a line by the letters at its extremities, the order of the letters is always that in which the line is to be drawn.

Vectors.—The expression $\overline{A\ B}$, in geometry, is merely the name of a line. Here it indicates the operation by which the line is drawn, that of carrying a tracing point in a certain direction for a certain distance. As indicating an operation, $\overline{A\ B}$ is called a Vector, and the operation is completely defined by the direction and distance of the transference. The starting point, which is called the origin of the vector, may be anywhere.

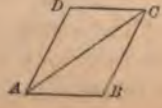
To define a finite straight line w must state its origin as well as it direction and length. All vectors, how ever, are regarded as equal which ar parallel (and drawn towards the sam parts) and of the same magnitude.

Any quantity, such, for instance, as velocity, or a force, which has a definit direction and a definite magnitude ma be treated as a vector, and may be ind cated in a diagram by a straight lin whose direction is parallel to the vecto and whose length represents, accordin to a determinate scale, the magnitude o the vector.

System of Three Particles.—Let u next consider a system of three parti cles.

Its configuration is represented by diagram of three points, A, B, C.

Fig. 1.



The position of B with respect to A is indicated by the vector $\overline{A\,B}$, and that of C with respect to B by the vector $\overline{B\,C}$.

It is manifest that from these data, when A is known, we can find B and then C, so that the configuration of the three points is completely determined.

The position of C with respect to A is indicated by the vector $\overline{A\,C}$, and by the last remark the value of $\overline{A\,C}$ must be deducible from those of $\overline{A\,B}$ and $\overline{B\,C}$.

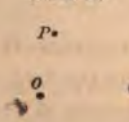
The result of the operation $\overline{A\,C}$ is to carry the tracing point from A to C. But the result is the same if the tracing point is carried first from A to B and then from B to C, and this is is the sum of the operations $\overline{A\,B}+\overline{B\,C}$.

Addition of Vectors.—Hence the rule for the addition of vectors may be stated thus:—From any point as origin draw the successive vectors in series, so that each vector begins at the end of the preceding one. The straight line from the origin to the extremity of the series represents the vector which is the sum of the vectors.

the position of its parts with respect to each other—remains the same, whatever point be chosen as origin. Many inquiries, however, are simplified by a proper selection of the origin.

Relative Position of Two Systems. —If the configurations of two different systems are known, each system having its own origin, and if we then wish to include both systems in a larger system, having, say, the same origin as the first of the two systems, we must ascertain the position of the origin of the second system with respect to that of the first, and we must be able to draw lines in the second system parallel to those in the first.

Fig. 2.



Then by "System of Three Particles," the position of a point P of the second

system, with respect to the first origin, O, is represented by the sum of the vector OP of that point with respect to the second origin, O′ and the vector OO′ of the second origin, O′ with respect to the first, O.

THREE DATA FOR THE COMPARISON OF TWO SYSTEMS.—We have an instance of this formation of a large system out of two or more smaller systems, when two neighboring nations, having each surveyed and mapped its own territory, agree to connect their surveys so as to include both countries in one system. For this purpose three things are necessary.

1st. A comparison of the origin selected by the one country with that selected by the other.

2d. A comparison of the directions of reference used in the two countries.

3d. A comparison of the standards of length used in the two countries.

1. In civilized countries latitude is always reckoned from the equator, but

We have described the method of combining several configurations into one system which includes them all. In this way we add to the small region, which we can explore by stretching our limbs, the more distant regions which we can reach by walking or by being carried. To these we add those of which we learn by the reports of others, and those inaccessible regions whose position we ascertain only by a process of calculation, till at last we recognize that every place has a definite position with respect to every other place, whether the one place is accessible from the other or not.

Thus from measurements made on the earth's surface we deduce the position of the center of the earth relative to known objects, and we calculate the number of cubic miles in the earth's volume quite independently of any hypothesis as to what may exist at the center of the earth, or in any other place beneath that thin layer of the crust of the earth which alone we can directly explore.

Error of Descartes.—It appears then, that the distance between one thing and another does not depend on any material thing between them, as Descartes seems to assert when he says (Princip. Phil., II. 18) that if that which is in a hollow vessel were taken out of it without anything entering to fill its place, the sides of the vessel, having nothing between them, would be in contact.

This assertion is grounded on the dogma of Descartes, that the extension in length, breadth, and depth which constitute space is the sole essential property of matter. "The nature of matter," he tells us, "or of body considered generally, does not consist in a thing being hard, or heavy, or colored, but only in its being extended in length, breadth, and depth" (Princip., II. 4). By thus confounding the properties of matter with those of space he arrives at the logical conclusion, that if the matter within a vessel could be entirely removed the space within the vessel would no

time during which I can work without feeling tired is greater or less now than when I first began to study. By our intercourse with other persons, and by our experience of natural processes which go on in a uniform or a rhythmical manner, we come to recognize the possibility of arranging a system of chronology in which all events whatever, whether relating to ourselves or to others, must find their place. Of any two events, say the actual disturbance at the star in Corona Borealis, which caused the luminous effects examined spectroscopically by Mr. Huggins on the 16th May, 1866, and the mental suggestion which first led Professor Adams or M. Leverrier to begin the researches which led to the discovery, by Dr. Galle, on the 23rd September, 1846, of the planet Neptune, the first named must have occured either before or after the other, or else at the same time.

Absolute, true, and mathematical Time is conceived by Newton as flowing at a constant rate, unaffected by the speed or

slowness of the motions of material things. It is also called duration. Relative, apparent, and common time is duration as estimated by the motion of bodies, as by days, months, and years. These measures of time may be regarded as provisional, for the progress of astronomy has taught us to measure the inequality in the lengths of days, months, and years, and thereby to reduce the apparent time to a more uniform scale, called Mean Solar Time.

Absolute Space.—Absolute space is conceived as remaining always similar to itself and immovable. The arrangement of the parts of space can no more be altered than the order of the portions of time. To conceive them to move from their places is to conceive a place to move away from itself.

But as there is nothing to distinguish one portion of time from another except by the different events which occur in them, so there is nothing to distinguish one part of space from another except its

it is possible for an event exactly simila
to occur at any other time and place.

There is another maxim which mus
not be confounded with that quoted a
the beginning of this article, whic
asserts "That like causes produce lik
effects."

This is only true when small varia
tions in the initial circumstances produc
only small variations in the final state o
the system. In a great many physica
phenomena this condition is satisfied
but there are other cases in which
small initial variation may produce
very great change in the final state o
the system, as when the displacement o
the points causes a railway train to ru
into another instead of keeping its pro
per course.

CHAPTER II.

ON MOTION.

DEFINITION OF DISPLACEMENT.—W
have already compared the position o
different points of a system at the sam

instant of time. We have next to compare the position of a point at a given instant with its position at a former instant, called the Epoch.

The vector which indicates the final position of a point with respect to its position at the epoch is called the Displacement of that point. Thus if A_1 is the initial and A_2 the final position of the point A, the line $\overline{A_1 A_2}$ is the displacement of A, and any vector $\overline{o\,a}$ drawn from the origin o parallel and equal to $\overline{A_1 A_2}$ indicates this displacement.

Diagram of Displacement. — If another point of the system is displaced from B_1 to B_2 the vector $\overline{o\,b}$ parallel and equal to $\overline{B_1 B_2}$ indicates the displacement of B.

In like manner the displacement of any number of points may be represented by vectors drawn from the same origin o. This system of vectors is called the Diagram of Displacement. It is not necessary to draw actual lines to repre-

system with respect to an external
are the same in direction and magn
the diagram of displacement is re
to two points—one corresponding
external point, and the other to
and every point of the displaced s
In this case the points of the syste
not displaced with respect to
another, but only with respect to t
ternal point.

This is the kind of displacement
occurs when a body of invariable
moves parallel to itself. It m
called uniform displacement.

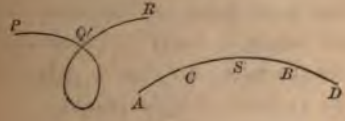
On Motion.—When the chan
configuration of a system is consi
with respect only to its state a
beginning and the end of the proc
change, and without reference t
time during which it takes place
called the displacement of the syst

When we turn our attention t
process of change itself, as taking
during a certain time and in a c
uous manner, the change of conf

tion is ascribed to the motion of the system.

On the Continuity of Motion.—When a material particle is displaced so as to pass from one position to another, it can only do so by traveling along some course or path from the one position to the other.

Fig. 4.



At any instant during the motion the particle will be found at some one point of the path, and if we select any point of the path, the particle will pass that point once at least* during its motion.

This is what is meant by saying that the particle described a continuous path.

* If the path cuts itself so as to form a loop, as P, Q, R, (fig. 4), the particle will pass the point of intersection, Q, twice, and if the particle returns on its own path, as in the path A, B, C, D, it may pass the same point, S, three or more times.

The motion of a material particle which has continuous existence in time and space is the type and exemplar of every form of continuity.

On Constant* Velocity.—If the motion of a particle is such that in equal intervals of time, however short, the displacements of the particle are equal and in the same direction, the particle is said to move with constant velocity.

It is manifest that in this case the path of the body will be a straight line, and the length of any part of the path will be proportional to the time of describing it.

The rate or speed of the motion is called the velocity of the particle, and its magnitude is expressed by saying that it is such a distance in such a time, as, for instance, ten miles an hour, or one meter per second. In general we select a unit of time, such as a second and measure

* When the successive values of a quantity for successive instances of time are equal, the quantity is said to be *constant*.

velocity by the distance described in unit of time.

If one meter be described in a second and if the velocity be constant, a thousandth or a millionth of a meter will be described in a thousandth or a millionth of a second. Hence, if we can observe or calculate the displacement during any interval of time, however short, we may deduce the distance which would be described in a longer time with the same velocity. This result, which enables us to state the velocity during the short interval of time, does not depend on the body's actually continuing to move at the same rate during the longer time. Thus we may know that a body is moving at the rate of ten miles an hour, though its motion at this rate may last for only the hundredth of a second.

On the Measurement of Velocity when Variable.—When the velocity of a particle is not constant, its value at any given instant is measured by the distance which would be described in

the method of deducing the rate of a
celeration from a knowledge of the tot
acceleration in any given time is pr
cisely analogous to that by which th
velocity at any instant is deduced from
knowledge of the displacement in an
given time.

The diagram of total accelerations co
structed for an interval of the nth pa
of the unit of time, and then magnifie
n times, is a diagram of the mean rat
of acceleration during that interval, an
by taking the interval smaller and smal
er, we ultimately arrive at the true rat
of acceleration at the middle of that in
terval.

As rates of acceleration have to b
considered in physical science muc
more frequently than total acceleration
the word acceleration has come to b
employed in the sense in which we hav
hitherto used the phrase—rate of ac
celeration.

In future, therefore, when we use th
word acceleration without qualificatio
we mean what we have here describe
as the rate of acceleration.

DIAGRAM OF ACCELERATIONS.—The diagram of accelerations is a system of points, each of which corresponds to one of the bodies of the material system, and is such that any line $\overline{\alpha\beta}$ in the diagram represents the rate of acceleration of the body B with respect to the body A.

It may be well to observe here that in the diagram of configuration we use the capital letters, A, B, C, &c., to indicate the relative position of the bodies of the system; in the diagram of velocities we use the small letters, a, b, c, to indicate the relative velocities of these bodies; and in the diagram of accelerations we use the Greek letters, α, β, γ, to indicate their relative accelerations.

ACCELERATION A RELATIVE TERM.—Acceleration, like position and velocity, is a relative term and cannot be interpreted absolutely.

If every particle of the material universe within the reach of our means of observation were at a given instant to have its velocity altered by compound-

the origin to the given point. direction of the mass-vector is the s as that of the vector of the mass, bu magnitude is the product of the r into the vector of the mass.

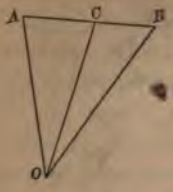
Thus if $\overline{OA}$ is the vector of the r A, the mass-vector is $\overline{OA}.A$.

Center of Mass of Two Partic —If A and B are two masses, and point C be taken in the straight line so that $\overline{BC}$ is to $\overline{CA}$ as A to B, then mass-vector of a mass A + B placed is equal to the sum of the mass-vec of A and B.

For $\overline{OA}.A+\overline{OB}.B=(\overline{OC}+\overline{CA})A+$
$+\overline{CB})B.$
$=\overline{OC}(A+B)+\overline{C}$
$+\overline{CB}.B.$

Now the mass-vectors $\overline{CA}.A$ and $\overline{C}$ are equal and opposite, and so des each other, so that $\overline{OA}.A+\overline{OB}.B=$ $(A+B)$

FIG. 7.



or, C is a point such that if the masses of A and B were concentrated at C, their mass-vector from any origin O would be the same as when A and B are in their actual positions. The point C is called the *Center of Mass* of A and B.

CENTER OF MASS OF A SYSTEM.—If the system consists of any number of particles, we may begin by finding the center of mass of any two particles, and substituting for the two particles a particle equal to their sum placed at their center of mass. We may then find the center of mass of this particle, together with the third particle of the system, and place the sum of the three particles at this point, and so on till we

have found the center of mass of th whole system.

The mass-vector drawn from any origin to a mass, equal to that of the whole system placed at the center of mass of the system, is equal to the sum of the mass-vectors drawn from the same origin to all the particles of the system.

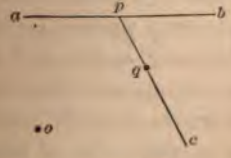
It follows from the proof in "Center of Mass of Two Particles," that the point found by the construction here given satisfies this condition. It is plain from the condition itself that only one point can satisfy it. Hence the construction must lead to the same result, as to the position of the center of mass, in whatever order we take the particles of the system.

The center of mass is therefore a definite point in the diagram of the configuration of the system. By assigning to the different points in the diagrams of displacement, velocity, total acceleration and rate of acceleration, the masses of the bodies to which they correspond, w

may find in each of these diagrams a point which corresponds to the center of mass, and indicates the displacement, velocity, total acceleration of the center of mass.

MOMENTUM REPRESENTED AS THE RATE OF CHANGE OF A MASS-VECTOR.—In the diagram of velocities, if the points o, a, b, c, correspond to the velocities of the origin O and the bodies A, B, C, and if p be the center of mass of A and B placed at a and b respectively, and if q is the center of mass of A+B placed at p and C at c, then q will be the center of mass of the system of bodies A, B, C, at a, b, c, respectively.

FIG. 8.



The velocity of A with respect to O is indicated by the vector $o\,a$, and that of

B and C by $\overline{o\,b}$ and $\overline{o\,c}$. $\overline{o\,p}$ is the velocity of the center of mass of A and B, and $\overline{o\,q}$ that of the center of mass of A, B, and C, with respect to O.

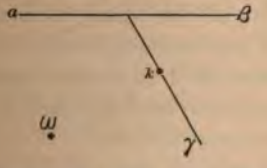
The momentum of A with respect to O is the product of the velocity into the mass, or $\overline{o\,a}$.A, or what we have already called the mass-vector, drawn from o to the mass A at a. Similarly the momentum of any other body is the mass-vector drawn from o to the point on the diagram of velocities corresponding to that body, and the momentum of the mass of the system concentrated at the center of mass is the mass-vector drawn from o to the whole mass at q.

Since, therefore, a mass-vector in the diagram of velocities is what we have already defined as a momentum, we may state the property proved in "Center of Mass of a System," in terms of momenta, thus: The momentum of a mass equal to that of the whole system, moving with the velocity of the center of mass of the system, is equal in magnitude and parallel in direction to the sum of the

momenta of all the particles of the system.

Effect of External Forces on the Motion of the Center of Mass.—In the same way in the diagram of Total Acceleration the vectors $\overline{\omega\,\alpha}$, $\overline{\omega\,\beta}$, drawn from the origin, represent the change of velocity of the bodies A, B, &c., during a certain interval of time. The corresponding mass-vectors, $\overline{\omega\,\alpha}$.A, $\overline{\omega\,\beta}$.B., &c., represent the corresponding changes

Fig. 9.



of momentum, or, by the second law of motion, the impulses of the forces acting on these bodies during that interval of time. If k is the center of mass of the system $\overline{\omega\,k}$ is the change of velocity during the interval, and $\overline{\omega\,k}$ (A+B+C) is the momentum generated in the mass

of motion, by defining that by the velocity of a body is meant the velocity of its center of mass. The body may be rotating, or it may consist of parts, and be capable of changes of configuration, so that the motions of different parts may be different, but we can still assert the laws of motion in the following form :

Law I. The center of mass of the system perseveres in its state of rest, or of uniform motion in a straight line, except in so far as it is made to change that state by forces acting on the system from without.

Law II. The change of momentum of the system during any interval of time is measured by the sum of the impulses of the external forces during that interval.

Method of Treating Systems of Molecules.—When the sytem is made up of parts which are so small that we cannot observe them, and whose motions are so rapid and so variable that even if we

could observe them we could not describe them, we are still able to deal with the motion of the center of mass of the system, because the internal forces which cause the variation of the motion of the parts do not affect the motion of the center of mass.

By the Introduction of the Idea of Mass we pass from Point-Vectors, Point Displacements, Velocities, Total Accelerations, and Rates of Acceleration, to Mass-Vectors, Mass Displacements, Momenta, Impulses, and Moving Forces.—In the diagram of rates of acceleration (Fig. 9, "Effect of External Forces on the Motion of the Center of Mass)," the vectors $\overline{\omega\alpha}$, $\overline{\omega\beta}$, &c., drawn from the origin, represent the rates of acceleration of the bodies A, B, &c., at a given instant, with respect to that of the origin O.

The corresponding mass-vectors, $\overline{\omega\alpha}.A$, $\overline{\omega\beta}.B$, &c., represent the forces acting on the bodies A, B, &c.

We sometimes speak of several forces

acting on a body, when the force acti
on the body arises from several differe
causes, so that we naturally consider t
parts of the force arising from th
different causes separately.

But when we consider force, not w
respect to its causes, but with respect
its effect—that of altering the motion
a body—we speak not of the forces, b
of the force acting on the body, and t
force is measured by the rate of chan
of the momentum of the body, and is
dicated by the mass-vector in the d
gram of rates of acceleration.

We have thus a series of differe
kinds of mass-vectors corresponding
the series of vectors which we ha
already discussed.

We have, in the first place, a syste
of mass-vectors with a common orig
which we may regard as a method
indicating the distribution of mass in
material system, just as the correspon
ing system of vectors indicate the g
metrical configuration of the system.

In the next place, by comparing t

distribution of mass at two different epochs, we obtain a system of mass-vectors of displacement.

The rate of mass displacement is momentum, just as the rate of displacement is velocity.

The change of momentum is impulse as the change of velocity is total acceleration.

The rate of change of momentum is moving force, as the rate of change of velocity is rate of acceleration.

Definition of a Mass-Area.—When a material particle moves from one point to another, twice the area swept out by the vector of the particle multiplied by the mass of the particle is called the mass-area of the displacement of the particle with respect to the origin from which the vector is drawn.

If the area is in one plane, the direction of the mass-area is normal to the plane, drawn so that, looking in the positive direction along the normal, the motion of the particle round its area ap-

pears to be the direction of the motion of the hands of a watch.

If the area is in one plane, the path of the particle must be divided into portions so small that each coincides sensibly with a straight line, and the mass-areas corresponding to these portions must be added together by the rule for the addition of vectors.

Angular Momentum.—The rate of change of a mass-area is twice the mass of the particle into the triangle, whose vertex is the origin and whose base is the velocity of the particle measured along the line through the particle in the direction of its motion. The direction of this mass-area is indicated by the normal drawn according to the rule given above.

The rate of change of the mass-area of a particle is called the Angular Momentum of the particle about the origin, and the sum of the angular momenta of all the particles is called the angular momentum of the system about the origin.

The angular momentum of a material system with respect to a point is, therefore, a quantity having a definite direction as well as a definite magnitude.

The definition of the angular momentum of a particle about a point may be expressed somewhat differently—as the product of the momentum of the particle with respect to that point into the perpendicular from that point on the line of motion of the particle at that instant.

MOMENT OF A FORCE ABOUT A POINT. —The rate of increase of the angular momentum of a particle is the continued product of the rate of acceleration of the velocity of the particle into the mass of the particle into the perpendicular from the origin on the line through the particle along which the acceleration takes place. In other words, it is the product of the moving force acting on the particle into the perpendicular from the origin on the line of action of this force.

Now the produce of a force into the perpendicular from the origin on its line

of action is called the Moment of the Force about the origin. The axis of the moment, which indicates its direction, is a vector drawn perpendicular to the plane passing through the force and the origin, and in such a direction that looking along this line in the direction in which it is drawn, the force tends to move the particle round the origin in the direction of the hands of a watch.

Hence the rate of change of the angular momentum of a particle about the origin is measured by the moment of the force which acts on the particle about that point.

The rate of change of the angular momentum of a material system about the origin is in like manner measured by the geometric sum of the moments of the forces which act on the particles of the system.

Conservation of Angular Momentum.—Now consider any two particles of the system. The forces acting on these two particles, arising from their

mutual action, are equal, opposite, and in the same straight line. Hence the moments of these forces about any point as origin are equal, opposite, and about the same axis. The sum of these moments is therefore zero. In like manner the mutual action between every other pair of particles in the system consists of two forces, the sum of whose moments is zero.

Hence the mutual action between the bodies of a material system does not affect the geometric sum of the moments of the forces. The only forces, therefore, which need be considered in finding the geometric sum of the moments are those which are external to the system—that is to say, between the whole or any part of the system and bodies not included in the system.

The rate of change of the angular momentum of the system is therefore measured by the geometric sum of the moments of the external forces acting on the system.

If the directions of all the external

forces pass through the origin, their moments are zero, and the angular momentum of the system will remain constant.

When a planet describes an orbit about the sun, the direction of the mutual action between the two bodies always passes through their common center of mass. Hence the angular momentum of either body about their common center of mass remains constant, so far as these two bodies only are concerned, though it may be affected by the action of other planets. If, however, we include all the planets in the system, the geometric sum of their angular momenta about their common center of mass will remain absolutely constant, whatever may be their mutual actions, provided no force arising from bodies external to the whole solar system acts in an unequal manner upon the different members of the system.

CHAPTER V.

ON WORK AND ENERGY.

DEFINITIONS.—Work *is the act of producing a change of configuration in a system in opposition to a force which resists that change.*

Energy *is the capacity of doing work.*

When the nature of a material system is such that if, after the system has undergone any series of changes, it is brought back in any manner to its original state, the whole work done by external agents on the system is equal to the whole work done by the system in overcoming external forces, the system is called a CONSERVATIVE SYSTEM.

PRINCIPLE OF CONSERVATION OF ENERGY.—The progress of physical science has led to the discovery and investigation of different forms of energy, and to the establishment of the doctrine that all material systems may be regarded as conservative systems, provided that all

the different forms of energy which exist in these systems are taken into account.

This doctrine, considered as a deduction from observation and experiment can, of course, assert no more than that no instance of a non-conservative system has hitherto been discovered.

As a scientific or science-producing doctrine, however, it is always acquiring additional credibility from the constantly increasing number of deductions which have been drawn from it, and which are found in all cases to be verified by experiment.

In fact the doctrine of the Conservation of Energy is the one generalized statement which is found to be consistent with fact, not in one physical science only, but in all.

When once apprehended it furnishes to the physical inquirer a principle on which he may hang every known law relating to physical actions, and by which he may be put in the way to discover the relations of such actions in new branches of science.

For such reasons the doctrine is commonly called the Principle of the Conservation of Energy.

General Statement of the Principle of the Conservation of Energy. —*The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.*

If, by the action of some agent external to the system, the configuration of the system is changed, while the forces of the system resist this change of configuration, the external agent is said to do work on the system. In this case the energy of the system is increased by the amount of work done on it by the external agent.

If, on the contrary, the forces of the system produce a change of configuration which is resisted by the external agent, the system is said to do work on the external agent, and the energy of the

system is diminished by the amount of work which it does.

Work, therefore, is a transference of energy from one system to another; the system which gives out energy is said to do work on the system which receives it, and the amount of energy given out by the first system is always exactly equal to that received by the second.

If, therefore, we include both systems in one larger system, the energy of the total system is neither increased nor diminished by the action of the one partial system on the other.

Measurement of Work.—Work done by an external agent on a material system may be described as a change in the configuration of the system taking place under the action of an external force tending to produce that change.

Thus, if one pound is lifted one foot from the ground by a man in opposition to the force of gravity, a certain amount of work is done by the man, and this quantity is known among engineers as one foot-pound.

Here the man is the external agent, the material system consists of the earth and the pound, the change of configuration is the increase of the distance between the matter of the earth and the matter of the pound, and the force is the upward force exerted by the man in lifting the pound, which is equal and opposite to the weight of the pound. To raise the pound a foot higher would, if gravity were a uniform force, require exactly the same amount of work. It is true that gravity is not really uniform, but diminishes as we ascend from the earth's surface, so that a foot-pound is not an accurately know quantity, unless we specify the intensity of gravity at the place. But for the purpose of illustration we may assume that gravity is uniform for a few feet of ascent, and in that case the work done in lifting a pound would be one foot-pound for every foot the pound is lifted.

To raise twenty pounds of water ten feet high requires 200 foot-pounds of work. To raise one pound ten feet high

requires ten foot-pounds, and as th
are twenty pounds the whole work
twenty times as much, or two hundr
foot-pounds.

The quantity of work done is, the
fore, proportional to the product of t
numbers representing the force exert
and the displacement in the direction
the force.

In the case of a foot-pound the for
is the weight of a pound—a quanti
which, as we know, is different in diff
ent places. The weight of a pound
pressed in absolute measure is numeric
ly equal to the intensity of gravity,
quantity denoted by g, the value
which in poundals to the pound va
from 32.227 at the pole to 32.117 at
equator, and diminishes without limit
we recede from the earth. In dynes
the gramme it varies from 978.1 to 98
Hence, in order to express work in a u
form and consistent manner, we m
multiply the number of foot-pounds
the number representing the intensity
gravity at the place. The work is th

reduced to foot-poundals. We shall always understand work to be measured in this manner and reckoned in foot-poundals when no other system of measurement is mentioned. When work is expressed in foot-pounds the system is that of *gravitation-measures*, which is not a complete system unless we also know the intensity of gravity at the place.

In the metrical system the unit of work is the Erg, which is the work done by a dyne acting through a centimeter. There are 421393.8 ergs in a foot-poundal.

Potential Energy.—The work done by a man in raising a heavy body is done in overcoming the attraction between the earth and that body. The energy of the material system, consisting of the earth and the heavy body, is thereby increased. If the heavy body is the leaden weight of a clock, the energy of the clock is increased by winding it up, so that the clock is able to go for a week in spite of the friction of the wheels and the resist-

ance of the air to the motion of the pendulum, and also to give out energy in other forms, such as the communication of the vibrations to the air, by which we hear the ticking of the clock.

When a man winds up a watch he does work in changing the form of the mainspring by coiling it up. The energy of the mainspring is thereby increased, so that as it uncoils itself it is able to keep the watch going.

In both these cases the energy communicated to the system depends upon a change of configuration.

KINETIC ENERGY.—But in a very important class of phenomena the work is done in changing the velocity of the body on which it acts. Let us take as a simple case that of a body moving without rotation under the action of a force. Let the mass of the body be M pounds, and let a force of F poundals act on it in the line of motion during an interval of time, T seconds. Let the velocity at the beginning of the interval be V and

that at the end V′ feet per second, and let the distance traveled by the body during the time be S feet. The original momentum is MV, and the final momentum is MV′, so that the increase of momentum is M (V′−V), and this, by the second law of motion is equal to FT, the *impulse* of the force F acting for the time T. Hence

$$FT=M\ (V'-V). \qquad (1)$$

Since the velocity increases uniformly with the time, the mean velocity is the arithmetical mean of the original and final velocities, or $\frac{1}{2}$ (V′+V).

We can also determine the mean velocity by dividing the space S by the time T, during which it is described.

$$\text{Hence } \frac{S}{T}=\tfrac{1}{2}(V'+V). \qquad (2)$$

Multiplying the corresponding members of equations (1) and (2) each by each we obtain—

$$FS=\tfrac{1}{2}\ MV'^2-\tfrac{1}{2}MV^2 \qquad (3)$$

Here FS is the work done by the force F acting on the body while it moves

through the space S in the direction of the force, and this is equal to the excess of $\frac{1}{2}$ MV'^{2} above $\frac{1}{2}$ MV^2. If we call $\frac{1}{2}$ MV^2, or half the product of the mass, into the square of the velocity, the *kinetic energy* of the body at first, then $\frac{1}{2}$ MV'^{2} will be the kinetic energy after the action of the force F through the space S. The energy is here expressed in foot-poundals.

We may now express the equation in words by saying that the work done by the force F in changing the motion of the body is measured by the increase of the kinetic energy of the body during the time that the force acts.

We have proved that this is true when the interval of time is so small that we may consider the force as constant during that time, and the mean velocity during the interval as the arithmetical mean of the velocities at the beginning and end of the interval. This assumption, which is exactly true when the force is constant, however long the interval may be, becomes in every case more and more

nearly true as the interval of time taken becomes smaller and smaller. By dividing the whole time of action into small parts, and proving that in each of these the work done is equal to the increase of the kinetic energy of the body, we may, by adding the successive portions of the work and the successive increments of energy, arrive at the result that the total work done by the force is equal to the total increase of kinetic energy.

If the force acts on the body in the direction opposite to its motion, the kinetic energy of the body will be diminished instead of being increased, and the force, instead of doing work on the body, will act as a resistance, which the body, in its motion, overcomes. Hence a moving body, as long as it is in motion, can do work in overcoming resistance, and the work done by the moving body is equal to the diminution of its kinetic energy, till at last, when the body is brought to rest, its kinetic energy is exhausted, and the whole work it has done is then equal

to the whole kinetic energy which it h
at first.

We now see the appropriateness
the name *kinetic energy*, which we ha
hitherto used merely as a name to den
the product $\frac{1}{2}$ MV^2. For the energy
a body has been defined as the capac
which it has of doing work, and it
measured by the work which it can
The *kinetic* energy of a body is t
energy it has in virtue of being in *moti*
and we have now shown that its value
expressed by $\frac{1}{2}$ MV^2 or $\frac{1}{2}$ MV$\times$V, th
is, half the product of its momentum in
its velocity.

Oblique Forces.—If the force acts
the body at right angles to the directi
of its motion it does not work on t
body, and it alters the direction but n
the magnitude of the velocity. T
kinetic energy, therefore, which depen
on the square of the velocity, remai
unchanged.

If the direction of the force is neith
coincident with, nor at right angles

at of the motion of the body we may solve the force into two components, ne of which is at right angles to the direction of motion, while the other is in ne direction of motion (or in the opposite direction).

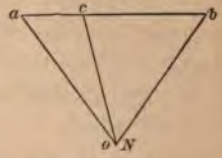
The first of these components may be left out of consideration in all calculations about energy, since it neither does work on the body nor alters its kinetic energy.

The second component is that which we have already considered. When it is in the direction of motion it increases the kinetic energy of the body by the amount of work which it does on the body. When it is in the opposite direction the kinetic energy of the body is diminished by the amount of work which the body does against the force.

Hence in all cases the increase of kinetic energy is equal to the work done on the body by external agency, and the diminution of kinetic energy is equal to the work done by the body against external resistance.

KINETIC ENERGY OF TWO PARTICLES REFERRED TO THEIR CENTER OF MASS. —The kinetic energy of a material system is equal to the kinetic energy of a mass equal to that of the system moving with the velocity of the center of mass of the system, together with the kinetic energy due to the motion of the parts of the system relative to its center of mass.

FIG. 10.



Let us begin with the case of two particles whose masses are A and B, and whose velocities are represented in the diagram of velocities by the lines $o\,a$ and $o\,b$. If c is the center of mass of a particle equal to A placed at a, and a particle equal to B placed at b, then $o\,c$ will represent the velocity of the center of mass of the two particles.

The kinetic energy of the system is the sum of the kinetic energies of the particles, or

$$T=\tfrac{1}{2}\,A\,\overline{oa}^2+\tfrac{1}{2}\,B\,\overline{ob}^2.$$

Expressing $\overline{oa}^2$ and $\overline{ob}^2$ in terms of $o\,c$, $c\,a$ and $c\,b$ and the angle $o\,c\,a=\theta$.

$$T=\tfrac{1}{2}\,A\,\overline{oc}^2+\tfrac{1}{2}\,A\,\overline{ca}^2-A\,o\,c.c\,a\cos\theta.$$
$$+\tfrac{1}{2}\,B\,\overline{oc}^2+\tfrac{1}{2}\,B\,\overline{cb}^2-B\,o\,c.c\,b\,cos\,\theta.$$

But since c is the center of mass of A at a, and B at b,

$$A.\,\overline{ca}+B\,\overline{cb}=0.$$

Hence adding

$$T=\tfrac{1}{2}\,(A+B)\,\overline{oc}^2+\tfrac{1}{2}\,A\,\overline{ca}^2+\tfrac{1}{2}\,B\,\overline{cb}^2,$$

or, the kinetic energy of the system of two particles A and B is equal to that of a mass equal to (A + B) moving with the velocity of the center of mass, together with that of the motion of the particles relative to the center of mass.

KINETIC ENERGY OF A MATERIAL SYSTEM REFERRED TO ITS CENTER OF MASS.—We have begun with the case of two

particles, because the motion of a particle is assumed to be that of its center of mass, and we have proved our proposition true for a system of two particles. But if the proposition is true for each of two material systems taken separately, it must be true of the system which they form together. For if we now suppose *oa* and *ob* to represent the velocities of the centers of mass of two material systems A and B, then *o c* will represent the velocity of the center of mass of the combined system A + B, and if T_A represents the kinetic energy of the motion of the system A relative to its own center of mass, and T_B the same for the system B, then if the proposition is true for the systems A and B taken separately, the kinetic energy of A is

$$\tfrac{1}{2} A \overline{oa}^2 + T_A,$$

and that of B

$$\tfrac{1}{2} B \overline{ob}^2 + T_B.$$

The kinetic energy of the whole is, therefore,

$$\tfrac{1}{2} A \overline{oa}^2 + \tfrac{1}{2} B \overline{ob}^2 + T_A + T_B,$$

or,

$$\tfrac{1}{2}(A+B)\overline{oc}^2+\tfrac{1}{2}A\overline{ca}^2+T_A+\tfrac{1}{2}B\overline{cb}^2+T_B.$$

The first term represents the kinetic energy of a mass equal to that of the whole system moving with the velocity of the center of mass of the whole system.

The second and third terms, taken together, represent the kinetic energy of the system A relative to the center of gravity of the whole system, and the fourth and fifth terms represent the same for the system B.

Hence if the proposition is true for the two systems A and B taken separately, it is true for the system compounded of A and B. But we have proved it true for the case of two particles; it is, therefore, true for three, four, or any other number of particles, and therefore for any material system.

The kinetic energy of a system referred to its center of mass is less than its kinetic energy when referred to any other point.

For the latter quantity exceeds the

former by a quantity equal to th kinetic energy of a mass equal to that o the whole system moving with th velocity of the center of mass relative t the other point, and since all kinetic energy is essentially positive, this excess must be positive.

AVAILABLE KINETIC ENERGY.—We have already seen in "The Motion of the Center of Mass," etc., that the mutual action between the parts of a material system cannot change the velocity of the center of mass of the system. Hence that part of the kinetic energy of the system which depends on the motion of the center of mass cannot be affected by any action internal to the system. It is therefore, impossible, by means of the mutual action of the parts of the system to convert this part of the energy into work. As far as the system itself is concerned, this energy is unavailable. It can be converted into work only by means of the action between this system and some other material system external to it.

Hence if we consider a material system unconnected with any other system, its available kinetic energy is that which is due to the motions of the parts of the system relative to its center of mass.

Let us suppose that the action between the parts of the system is such that after a certain time the configuration of the system becomes invariable, and let us call this process the solidification of the system. We have shown that the angular momentum of the whole system is not changed by any mutual action of its parts. Hence if the original angular momentum is zero, the system, when its form becomes invariable, will not rotate about its center of mass, but if it moves at all will move parallel to itself, and the parts will be at rest relative to the center of mass. In this case, therefore, the whole available energy will be converted into work by the mutual action of the parts during the solidification of the system.

If the system has angular momentum, it will have the same angular momentum

when solidified. It will therefore rotate about its center of mass, and will therefore still have energy of motion relative to its center of mass, and this remaining kinetic energy has not been converted into work.

But if the parts of the system are allowed to separate from one another in directions perpendicular to the axis of the angular momentum of the system, and if the system when thus expanded is solidified, the remaining kinetic energy of rotation round the center of mass will be less and less the greater the expansion of the system, so that by sufficiently expanding the system we may make the remaining kinetic energy as small as we please, so that the whole kinetic energy relative to the center of mass of the system may be converted into work within the system.

Potential Energy.—The potential energy of a material system is the capacity which it has of doing work depending on other circumstances than the

motion of the system. In other words, potential energy is that energy which is not kinetic.

In the theoretical material system which we build up in our imagination from the fundamental ideas of matter and motion, there are no other conditions present except the configuration and motion of the different masses of which the system is composed. Hence in such a system the circumstances upon which the energy must depend are motion and configuration only, so that, as the kinetic energy depends on the motion, the potential energy must depend on the configuration.

In many real material systems we know that part of the energy does depend on the configuration. Thus the mainspring of a watch has more energy when coiled up than when partially uncoiled, and two bar magnets have more energy when placed side by side with their similar poles turned the same way than when their dissimilar poles are placed next each other.

Elasticity.—In the case of the spring we may trace the connection between the coiling of the spring and the force which it exerts somewhat further by conceiving the spring divided (in imagination) into very small parts or elements. When the spring is coiled up, the form of each of these small parts is altered, and such an alteration of the form of a solid body is called a Strain.

In solid bodies strain is accompanied with internal force or stress; those bodies in which the stress depends simply on the strain are called Elastic, and the property of exerting stress when strained is called Elasticity.

We thus find that the coiling of the spring involves the strain of its elements, and that the external force which the spring exerts is the resultant of the stresses in its elements.

We thus substitute for the immediate relation between the coiling of the spring and the force which it exerts a relation between the strains and stresses of the elements of the spring; that is to say, for

a single displacement and a single force, the relation between which may in some cases be of an exceedingly complicated nature, we substitute a multitude of strains and an equal number of stresses, each strain being connected with its corresponding stress by a much more simple relation.

But when all is done, the nature of the connection between configuration and force remains as mysterious as ever. We can only admit the fact, and if we call all such phenomena phenomena of elasticity, we may find it very convenient to classify them in this way, provided we remember that by the use of the word elasticity we do not profess to explain the cause of the connection between configuration and energy.

Action at a Distance.—In the case of the two magnets there is no visible substance connecting the bodies between which the stress exists. The space between the magnets may be filled with air or with water, or we may place the

magnets in a vessel and remove the air by an air-pump, till the magnets are left in what is commonly called a vacuum, and yet the mutual action of the magnets will not be altered. We may even place a solid plate of glass or metal or wood between the magnets, and still we find that their mutual action depends simply on their relative position, and is not perceptibly modified by placing any substance between them, unless that substance is one of the magnetic metals. Hence the action between the magnets is commonly spoken of as *action at a distance.*

Attempts have been made, with a certain amount of success,* to analyze this action at a distance into a continuous distribution of stress in an invisible medium, and thus to establish an analogy between the magnetic action and the action of a spring or a rope in transmitting force; but still the general fact that strains or changes of configuration are

* See Clerk Maxwell's "Treatise on Electricity and Magnetism," Vol. II., Art. 641.

accompanied by stresses or internal forces, and that thereby energy is stored up in the system so strained, remains an ultimate fact which has not yet been explained as the result of any more fundamental principle.

Theory of Potential Energy more Complicated than that of Kinetic Energy.—Admitting that the energy of a material system may depend on its configuration, the mode in which it so depends may be much more complicated than the mode in which the kinetic energy depends on the motion of the system. For the kinetic energy may be calculated from the motion of the parts of the system by an invariable method. We multiply the mass of each part by half the square of its velocity, and take the sum of all such products. But the potential energy arising from the mutual action of two parts of the system may depend on the relative position of the parts in a manner which may be different in different instances. Thus when two

billiard balls approach each other from a distance, there is no sensible action between them till they come so near one another that certain parts appear to be in contact. To bring the centers of the two balls nearer, the parts in contact must be made to yield, and this requires the expenditure of work.

Hence in this case the potential energy is constant for all distances greater than the distance of first contact, and then rapidly increases when the distance is diminished.

The force between magnets varies with the distance in a very different manner, and in fact we find that it is only by experiment that we can ascertain the form of the relation between the configuration of a system and its potential energy.

APPLICATION OF THE METHOD OF ENERGY TO THE CALCULATION OF FORCES. —A complete knowledge of the mode in which the energy of a material system varies when the configuration and motion

of the system are made to vary is mathematically equivalent to a knowledge of all the dynamical properties of the system. The mathematical methods by which all the forces and stresses in a moving system are deduced from the single mathematical formula which expresses the energy as a function of the variables have been developed by Lagrange, Hamilton, and other eminent mathematicians, but it would be difficult even to describe them in terms of the elementary ideas to which we restrict ourselves in this book. An outline of these methods is given in my treatise on Electricity, Part IV., Chapter V., Article 553, and the application of these dynamical methods to electro-magnetic phenomena is given in the chapters immediately following.

But if we consider only the case of a system at rest it is easy to see how we can ascertain the forces of the system when we know how its energy depends on its configuration.

For *let us* suppose that an agent ex-

ternal to the system produces a displ
ment from one configuration to anot
then if in the new configuration the
tem possess more energy than it di
first, it can have received this incr
of energy only from the external ag
This agent must therefore have don
amount of work equal to the incr
of energy. It must therefore have
erted force in the direction of the
placement, and the mean value of
force, multiplied into the displacem
must be equal to the work done. He
the mean value of the force may
found by dividing the increase of ene
by the displacement.

If the displacement is large this fo
may vary considerably during the
placement, so that it may be difficult
calculate its mean value; but since
force depends on the configuration, if
make the displacement smaller
smaller the variation of the force
become smaller and smaller, so that
last the force may be regarded as sensi
constant during the displacement.

If, therefore, we calculate for a given configuration the *rate* at which the energy increases with the displacement, by a method similar to that described "On the Measurement of Velocity when Variable;" "Diagram of Velocities," and "On the Rate of Acceleration," this rate will be numerically equal to the force exerted by the external agent in the direction of the displacement.

If the energy diminishes instead of increasing as the displacement increases, the system must do work on the external agent, and the force exerted by the external agent must be in the direction opposite to that of displacement.

Specification of the Direction of Forces.—In treatises on dynamics the forces spoken of are usually those exerted by the external agent on the material system. In treatises on electricity, on the other hand, the forces spoken of are usually those exerted by the electrified system against an external agent which prevents the system from moving. It is

necessary, therefore, in reading any statement about forces, to ascertain whether the force spoken of is to be regarded from the one point of view or the other.

We may in general avoid any ambiguity by viewing the phenomenon as a whole, and speaking of it as a stress exerted between two points or bodies, and distinguishing it as a tension or a pressure, an attraction or a repulsion, according to its direction, "Action and Reaction are the Partial Aspects of a Stress."

Application to a System in Motion. —It thus appears that from a knowledge of the potential energy of a system in every possible configuration we may deduce all the external forces which are required to keep the system in that configuration. If the system is at rest, and if these external forces are the actual forces, the system will remain in equilibrium. If the system is in motion the force acting on each particle is that arising from the connections of the sys-

tem (equal and opposite to the external force just calculated), together with any external force which may be applied to it. Hence a complete knowledge of the mode in which the potential energy varies with the configuration would enable us to predict every possible motion of the system under the action of given external forces, provided we were able to overcome the purely mathematical difficulties of the calculation.

Application of the Method of Energy to the Investigation of Real Bodies.—When we pass from abstract dynamics to physics—from material systems, whose only properties are those expressed by their definitions, to real bodies, whose properties we have to investigate—we find that there are many phenomena which we are not able to explain as changes in the configuration and motion of a material system.

Of course if we begin by assuming that the real bodies are systems composed of matter which agrees in all re-

spects with the definitions we have laid down, we may go on to assert that all phenomena are changes of configuration and motion, though we are not prepared to define the kind of configuration and motion by which the particular phenomena are to be explained. But in accurate science such asserted explanations must be estimated, not by their promises, but by their performances. The configuration and motion of a system are facts capable of being described in an accurate manner, and therefore, in order that the explanation of a phenomenon by the configuration and motion of a material system may be admitted as an addition to our scientific knowledge, the configurations, motions, and forces must be specified, and shown to be consistent with known facts, as well as capable of accounting for the phenomenon.

VARIABLES ON WHICH THE ENERGY DEPENDS.—But even when the phenomena we are studying have not yet been explained dynamically, we are still

able to make great use of the principle of the conservation of energy as a guide to our researches.

To apply this principle, we in the first place assume that the quantity of energy in a material system depends on the state of that system, so that for a given state there is a definite amount of energy.

Hence the first step is to define the different states of the system, and when we have to deal with real bodies we must define their state with respect not only to the configuration and motion of their visible parts, but if we have reason to suspect that the configuration and motion of their invisible particles influence the visible phenomenon, we must devise some method of estimating the energy thence arising.

Thus pressure, temperature, electric potential, and chemical composition are variable quantities, the values of which serve to specify the state of a body, and in general the energy of the body depends on the values of these and other variables.

ENERGY IN TERMS OF THE VARIABLES. —The next step in our investigation is to determine how much work must be done by external agency on the body in order to make it pass from one specified state to another.

For this purpose it is sufficient to know the work required to make the body pass from a particular state, which we may call the *standard state*, into any other specified state. The energy in the latter state is equal to that in the standard state, together with the work required to bring it from the standard state into the specified state. The fact that this work is the same through whatever series of states the system has passed from the standard state to the specified state is the foundation of the whole theory of energy.

Since all the phenomena depend on the variations of the energy of the body, and not on its total value, it is unnecessary, even if it were possible, to form any estimate of the energy of the body *in* its standard state.

Y OF HEAT.—One of the most
t applications of the principle
nservation of energy is to the
tion of the nature of heat.
time it was supposed that the
e between the states of a body
and when cold was due to the
of a substance called caloric,
isted in greater abundance in
when hot than when cold. But
iments of Rumford on the heat
by the friction of metal, and of
the melting of ice by friction,
wn that when work is spent in
ng friction, the amount of heat
is proportional to the work

xperiments of Hirn have also
at when heat is made to do work
m-engine, part of the heat dis-
and that the heat which disap-
roportional to the work done.
y careful measurement of the
ent in friction, and of the heat
, has been made by Joule, who
t *the heat* required to raise one

ıe energy given out in the form of
ıanical work may be measured di-
y, or it may be transformed into
by friction. The energy spent in
ucing the electric current may be
ıated as heat by causing the current
ow through a conductor of such a
that the heat generated in it can
y be measured. Care must be taken
no energy is transmitted to a
ınce in the form of sound or radiant
without being duly accounted for.
ıe energy remaining in the mixture,
ther with the energy which has
ped, must be equal to the original
gy.

ndrews, Favre and Silbermann, and
rs, have measured the quantity of
produced when a certain quantity
xygen or of chlorine combines with
quivalent of other substances. These
surements enable us to calculate the
ss of the energy which the substances
erned had in their original state,
ı uncombined, above that which they
after combination.

IFIC WORK TO BE DONE.—
a great deal of excellent work
kind has already been done, the
of the field hitherto investigated
s quite insignificant when we con-
the boundless variety and com-
y of the natural bodies with which
ave to deal.

fact the special work which lies be-
the physical inquirer, in the present
e of science, is the determination of
quantity of energy which enters or
ves a material system during the pas-
ge of the system from its standard
te to any other definite state.

HISTORY OF THE DOCTRINE OF ENERGY.
The scientific importance of giving a
me to the quantity which we call kine-
energy seems to have been first rec-
nized by Leibnitz, who gave to the
oduct of the mass by the square of the
ocity the name of *Vis Viva*. This is
ce the kinetic energy.

Newton, in the "Scholium to the
ws of Motion," expresses the relation

between the rate at which work is
by the external agent, and the ra
which it is given out, stored u
transformed by any machine or
material system, in the following s
ment, which he makes in order to s
the wide extent of the application o
Third Law of Motion.

"If the action of the external age
estimated by the product of its force
its velocity, and the reaction of th
sistance in the same way by the pro
of the velocity of each part of the sy
into the resisting force arising
friction, cohesion, weight, and acce
tion, the action and reaction wil
equal to each other, whatever be th
ture and motion of the system."
this statement of Newton's impli
contains nearly the whole doctri
energy was first pointed out by Tho
and Tait.

The words Action and Reactio
they occur in the enunciation of
Third Law of Motion are explaine
mean Forces, that is to say, they ar

opposite aspects of one and the same Stress.

In the passage quoted above a new and different sense is given to these words by estimating Action and Reaction by the product of a force into the velocity of its point of application. According to this definition the Action of the external agent is the rate at which it does work. This is what is meant by the Power of a steam-engine or other prime mover. It is generally expressed by the estimated number of ideal horses which would be required to do the work at the same rate as the engine, and this is called the Horse-power of the engine.

When we wish to express by a single word the rate at which work is done by an agent we shall call it the Power of the agent, defining the power as the work done in the unit of time.

The use of the term Energy, in a precise and scientific sense, to express the quantity of work which a material sys-

tem can do, was introduced by Dr. Young.*

ON THE DIFFERENT FORMS OF ENERGY. —The energy which a body has in virtue of its motion is called kinetic energy.

A system may also have energy in virtue of its configuration, if the forces of the system are such that the system will do work against external resistance while it passes into another configuration. This energy is called Potential Energy. Thus when a stone has been lifted to a certain height above the earth's surface, the system of two bodies, the stone and the earth, has potential energy, and is able to do a certain amount of work during the descent of the stone. This potential energy is due to the fact that the stone and the earth attract each other, so that work has to be spent by the man who lifts the stone and draws it away from the earth, and after the stone is lifted the attraction between the earth and the stone is capable of doing

* "Lectures on Natural Philosophy," Lecture VIII.

work as the stone descends. This kind of energy, therefore, depends upon the work which the forces of the system would do if the parts of the system were to yield to the action of these forces. This is called the "Sum of the Tensions" by Helmholtz in his celebrated memoir on the "Conservation of Energy."* Thomson called it Statical Energy; it has also been called Energy of Position; but Rankine introduced the term Potential Energy—a very felicitous expression, since it not only signifies the energy which the system has not in actual possession, but only has the power to acquire, but it also indicates its connection with what has been called (on other grounds) the Potential Function.

The different forms in which energy has been found to exist in material systems have been placed in one or other of these two classes—Kinetic Energy, due to motion, and Potential Energy, due to configuration.

* Berlin, 1847. Translated in Taylor's "Scientific Memoirs," Feb. 1853.

But this may be regarded as a
application of the definition of a
and not as a new physical truth.

It is when we come to define
forces as those which produce equal
of acceleration in the same mass,
equal masses are those which are eq
accelerated by equal forces, that we
that these definitions of equality am
to the assertion of the physical t
that the comparison of quantitie
matter by the forces required to pro
in them a given acceleration is a me
which always leads to consistent re
whatever be the absolute values o
forces and the accelerations.

Stress.—The next step in the sc
of force is that in which we pass
the consideration of a force as actin
a body, to that of its being one a
of that mutual action between
bodies, which is called by Newton A
and Reaction, and which is now
briefly expressed by the single
Stress.

Relativity of Dynamical Knowledge.—Our whole progress up to this point may be described as a gradual development of the doctrine of relativity of all physical phenomena. Position we must evidently acknowledge to be relative, for we cannot describe the position of a body in any terms which do not express relation. The ordinary language about motion and rest does not so completely exclude the notion of their being measured absolutely, but the reason of this is, that in our ordinary language we tacitly assume that the earth is at rest.

As our ideas of space and motion become clearer, we come to see how the whole body of dynamical doctrine hangs together in one consistent system.

Our primitive notion may have been that to know absolutely where we are, and in what direction we are going, are essential elements of our knowledge as conscious beings.

But this notion, though undoubtedly held by many wise men in ancient times, has been gradually dispelled from the minds of students of physics.

There are no landmarks in spa
portion of space is exactly like
other portion, so that we cann
where we are. We are, as it w
an unruffled sea, without stars, c
soundings, wind, or tide, and we
tell in what direction we are going
have no log which we can cast
take a dead reckoning by; we ma
pute our rate of motion with res
the neighboring bodies, but we
know how these bodies may be
in space.

Relativity of Force.—We
even tell what force may be act
us; we can only tell the differe
tween the force acting on one thi
that acting on another.

We have an actual example of
our every-day experience. The
moves round the sun in a year a
tance of 91,520,000 miles, or 1.47
centimeters. It follows from this
force is exerted on the earth in th
tion of the sun, which produces

celeration of the earth in the direction of the sun of about 0.019 in feet and seconds, or about $\frac{1}{1680}$ of the intensity of gravity at the earth's surface.

A force equal to the sixteen-hundredth part of the weight of a body might be easily measured by known experimental methods, especially if the direction of this force were differently inclined to the vertical at different hours of the day.

Now, if the attraction of the sun were exerted upon the solid part of the earth, as distinguished from the movable bodies on which we experiment, a body suspended by a string, and moving with the earth, would indicate the difference between the solar action on the body, and that on the earth as a whole.

If, for example, the sun attracted the earth and not the suspended body, then at sunrise the point of suspension, which is rigidly connected with the earth, would be drawn towards the sun, while the suspended body would be acted on only by the earth's attraction, and the

string would appear to be deflec
away from the sun by a sixteen-ht
redth part of the length of the str
At sunset the string would be deflec
away from the setting sun by an eq
amount; and as the sun sets at a differ
point of the compass from that at wh
he rises the deflections of the str
would be in different directions, and
difference in the position of the plu
line at sunrise and sunset would
easily observed.

But instead of this, the attraction
gravitation is exerted upon all kinds
matter equally at the same distance fr
the attracting body. At sunrise
sunset the center of the earth and
suspended body are nearly at the sa
distance from the sun, and no deflect
of the plumb-line due to the sun's
traction can be observed at these tim
The attraction of the sun, therefore,
so far as it is exerted equally upon
bodies on the earth, produces no eff
on their relative motions. It is only
differences of the intensity and direct

of the attraction acting on different parts of the earth which can produce any effect, and these differences are so small for bodies at moderate distances that it is only when the body acted on is very large, as in the case of the ocean, that their effect becomes perceptible in the form of tides.

ROTATION.—In what we have hitherto said about the motion of bodies, we have tacitly assumed that, in comparing one configuration of the system with another, we are able to draw a line in the final configuration parallel to a line in the original configuration. In other words, we assume that there are certain directions in space which may be regarded as constant, and to which other directions may be referred during the motion of the system.

In astronomy, a line drawn from the earth to a star may be considered as fixed in direction, because the relative motion of the earth and the star is in general so small compared with the distance be-

The distances between the bodies composing the universe, whether celestial or terrestrial, and the angles between the lines joining them, are all that can be ascertained without an appeal to dynamical principles, and these will not be affected if any motion of rotation of the whole system, similar to that of a rigid body about an axis, is combined with the actual motion; so that from a geometrical point of view the Copernican system, according to which the earth rotates, has no advantage, except that of simplicity, over that in which the earth is supposed to be at rest, and the apparent motions of the heavenly bodies to be their absolute motions.

Even if we go a step further, and consider the dynamical theory of the earth rotating round its axis, we may account for its oblate figure, and for the equilibrium of the ocean and of all other bodies on its surface on either of two hypothesis—that of the motion of the earth round its axis, or that of the earth not rotating, but caused to assume its

ters or leaves the system that we take any account of it.

We must, therefore, regard the ene of a material system as a quantity which we may ascertain the increase diminution as the system passes fr one definite condition to another. absolute value of the energy in standard condition is unknown to and it would be of no value to us if did know it, as all phenomena depend the variations of the energy, and not its absolute value.

A Complete Discussion of Ener would include the whole of Physi Science.—The discussion of the vari forms of energy—gravitational, elect magnetic, molecular, thermal, &c.—w the conditions of the transference energy from one form to another, the constant dissipation of the ene available for producing work, constitu the whole of physical science, in so as it has been developed in the dyna cal form under the various designati

of Astronomy, Electricity, Magnetism, Optics, Theory of the Physical States of Bodies, Thermo-dynamics, and Chemistry.

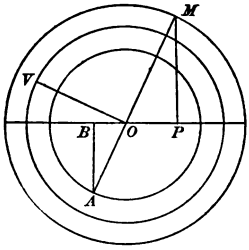
CHAPTER VII.

THE PENDULUM AND GRAVITY.

ON UNIFORM MOTION IN A CIRCLE.—Let M (Fig. 11) be a body moving in a circle with velocity V.

Let $\overline{O M} = r$ be the radius of the circle.

FIG. 11.



The direction of the velocity of M is that of the tangent to the circle. Draw

The force acting on the body M is equal to the product of this acceleration into the mass of the body, or if F be this force

$$F = \frac{MV^2}{r}$$

Centrifugal Force.—This is the force which must act on the body M in order to keep it in the circle of radius v, in which it is moving with velocity V.

The direction of this force is towards the center of the circle.

If this force is applied by means of a string fastened to the body, the string will be in a state of tension. To a person holding the other end of the string this tension will appear to be directed towards the body M, as if the body M had a tendency to move away from the center of the circle which it is describing.

Hence this latter force is often called Centrifugal Force.

The force which really acts on the body, *being directed* towards the center

O, and whose radius is the leng
string, which we shall denote by

FIG. 12.

Let O (Fig. 12) be the point of s
pension and O A the position of t
pendulum when hanging vertical
When the bob is at M it is higher th
when it is at A by the height $AP = \frac{\overline{A}}{\overline{A}}$
where AM is the chord of the arc AI
and $AB = 2l$.

If M be the mass of the bob and g t
intensity of gravity the weight of t
bob will be Mg and the work do
against gravity during the motion of t

b from A to M will be $Mg\,\overline{AP}$. This, erefore, is the potential energy of the ndulum when the bob is at M, reckon- g the energy zero when the bob is at .

We may write this energy

$$\frac{Mg}{2l}\,\overline{AM}^2$$

The potential energy of the bob when isplaced through any arc varies as the quare of the chord of that arc.

If it had varied as the square of the rc itself in which the bob moves, the ibrations would have been strictly sochronous. As the potential energy aries more slowly than the square of the arc, the period of each vibration will be greater when the amplitude is greater.

For very small vibrations, however, we may neglect the difference between the chord and the arc, and denoting the arc by x we may write the potential energy

$$\frac{Mg}{2l}x^2$$

Thus the actual observation co
not in watching the beginning an
of one vibration, but in determinin
duration of a series of many hu
vibrations, and thence deducing the
of a single vibration.

The observer is relieved from the
of counting the whole number of
tions, and the measurement is mad
of the most accurate in the whole
of practical science by the follo
method:

Method of Observation.—A p
lum clock is placed behind the e
mental pendulum, so that when
pendulums are hanging vertically
bob, or some other part of the e
mental pendulum, just hides a
spot on the clock pendulum, as se
a telescope fixed at some distan
front of the clock.

Observations of the transit of "
stars" across the meridian are
from time to time, and from thes
rate of the clock is deduced in ter
"mean solar time."

The experimental pendulum is then
a swinging, and the two pendulums
e observed through the telescope.
et us suppose that the time of a single
bration is not exactly that of the
ock pendulum, but a little more.

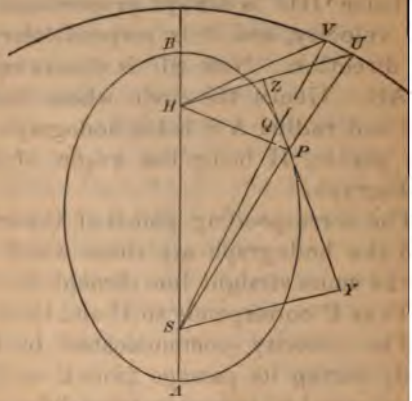
The observer at the telescope sees the
lock pendulum always gaining on the
xperimental pendulum, till at last the
xperimental pendulum just hides the
hite spot on the clock pendulum as it
rosses the vertical line. The time at
hich this takes place is observed and
ecorded as the First Positive Coinci-
ence.

The clock pendulum continues to gain
n the other, and after a certain time
he two pendulums cross the vertical
ne at the same instant in opposite
irections. The time of this is recorded
s the First Negative Coincidence.
fter an equal interval of time there
ill be a second positive coincidence,
nd so on.

By this method the clock itself counts
he number, N, of vibrations of its own

H U will be proportional and perpe
cular to the velocity at P.

FIG. 16.



For bisect H U in Z and join Z P,
will be a tangent to the ellipse at
let S Y be a perpendicular from S
this tangent.

If v is the velocity at P, and h tw
the area swept out in unit of time h
$\overline{SY}$.

Also if b is half the conjugate axis
the ellipse

$$\overline{SY} \cdot \overline{HZ} = b^2$$

Now HU = 2HZ; hence

$$v = \tfrac{1}{2}\frac{h}{b^2}\overline{\mathrm{HU}}$$

Hence HU is always proportional to the velocity, and it is perpendicular to its direction. Now SU is always equal to AB. Hence the circle whose center is S and radius A B is the hodograph of the planet, H being the origin of the hodograph.

The corresponding points of the orbit and the hodograph are those which lie in the same straight line through S.

Thus P corresponds to U and Q to V.

The velocity communicated to the body during its passage from P to Q is represented by the geometrical difference between the vectors HU and HV, that is, by the line UV, and it is perpendicular to this arc of the circle, and is therefore, as we have already proved, directed towards S.

If P Q is the arc described in unit of time, then UV represents the acceleration, and since UV is on a circle whose center is S, UV will be a measure of the

angular velocity of the planet about S. Hence the acceleration is proportional to the angular velocity, and this by "Angular Velocity" is inversely as the square of the distance SP. Hence the acceleration of the planet is in the direction of the sun, and is inversely as the square of the distance from the sun.

This, therefore, is the law according to which the attraction of the sun on a planet varies as the planet moves in its orbit and alters its distance from the sun.

Force on a Planet.—Since, as we have already shown, the orbit of the planet with respect to the center of mass of the sun and planet has its dimensions in the ratio of S to S+P to those of the orbit of the planet with respect to the Sun, if $2a$ and $2b$ are the axes of the orbit of the planet with respect to the sun, the area is $\pi\ a\ b$, and if T is the time of going completely round the orbit, the value of h is $2\pi \frac{ab}{T}$

The velocity with respect to the sun is, therefore,

$$\pi \frac{a}{T\,b} \overline{HU}$$

With respect to the center of mass it is

$$\frac{S}{S+P} \frac{\pi a}{T b} \overline{HU}$$

The acceleration of the planet towards the center of mass is

$$\frac{S}{S+P} \frac{\pi a}{T b} U\,V$$

and the impulse on that planet whose mass is P is therefore

$$\frac{S \,.\, P}{S+P} \frac{\pi a}{T b} U\,V$$

Let t be the time of describing P Q, then twice the area S P Q is

$$h\,t = \omega\, r^2\, t$$

and $$UV = 2a\ \omega t = 2a \frac{h}{r^2}\, t = 4\pi \frac{a^2 b}{T r^2} t.$$

Hence the force on the planet is

$$F = 4\pi^2 \frac{S \,.\, P}{S+P} \frac{a^3}{T^2 r^2}$$

This then is the value of the stress or

"Queries" printed at the end of his "Opticks."

The attempts which have been made since the time of Newton to solve this difficult question are few in number, and have not led to any well-established result.

Application of Newton's Method of Investigation.—The method of investigating the forces which act between bodies which was thus pointed out and exemplified by Newton in the case of the heavenly bodies, was followed out successfully in the case of electrified and magnetized bodies by Cavendish, Coulomb, and Poisson.

The investigation of the mode in which the minute particles of bodies act on each other is rendered more difficult from the fact that both the bodies we consider and their distances are so small that we cannot perceive or measure them, and we are therefore unable to observe their motions as we do those of

planets, or of electrified and magnetized bodies.

METHODS OF MOLECULAR INVESTIGATIONS.—Hence the investigations of molecular science have proceeded for the most part by the method of hypothesis, and comparison of the results of the hypothesis with the observed facts.

The success of this method depends on the generality of the hypothesis we begin with. If our hypothesis is the extremely general one that the phenomena to be investigated depend on the configuration and motion of a material system, then if we are able to deduce any available results from such an hypothesis, we may safely apply them to the phenomena before us.

If, on the other hand, we frame the hypothesis that the configuration, motion, or action of the material system is of a certain definite kind, and if the results of this hypothesis agree with the phenomena, then, unless we can prove